The *k*-function: Continued Fractions and Platonic Solids (Part II)

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Dedicated to George Lucas for a great Episode III.

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I. Introduction

In his first letter to G.H. Hardy, Ramanujan gave the beautiful continued fraction,

\[ \sqrt{\phi + 2} - \phi = \frac{e^{-2\pi/5}}{1 + \frac{e^{-2\pi}}{1 + \frac{e^{-4\pi}}{1 + \frac{e^{-6\pi}}{1 + \ldots}}}} \]

where \( \phi = (1 + \sqrt{5})/2 \) is the golden ratio and which suddenly appears in this fraction as if from nowhere. In his second letter, he gave another example in the same vein, though for the purposes of this paper, we will give a simpler one,

\[ x = \frac{e^{-4\pi/5}}{1 + \frac{e^{-4\pi}}{1 + \frac{e^{-8\pi}}{1 + \frac{e^{-12\pi}}{1 + \ldots}}}} \]

where \( x = 0.0810023\ldots \) is the appropriate root of the equation,

\[ x^{-1} - x = 1 + 5^{3/4} \phi + 5^{1/2} \phi^2 \]

In his second Notebook, Ramanujan noted down the general form of another kind of continued fraction. We can provide a specific example, given by,
\[ \sqrt{-1 + \sqrt{2}} = \frac{\sqrt{2e^{-2\pi / \sqrt{2}}}}{e^{-2\pi}} \]
\[ 1 + \frac{e^{-4\pi}}{1 + e^{-2\pi} + \frac{e^{-6\pi}}{1 + e^{-5\pi} + \ldots}} \]

So how in the world do we find the evaluation of beautiful continued fractions like these? It turns out that one answer to that question is by finding a modular relation between the \( j \)-function, \( j(t) \), and certain quotients of the Dedekind eta function, \( \eta(t) \). This is related to Part I of this paper, since we mentioned previously that an explicit modular relation of order \( p \) with rational coefficients can be found between the \( j \)-function and a Hauptmodul whenever \( p-1 \) divides 24. There we discussed the prime orders \( p = 2, 3, 5, 7, 13 \). In Part II, we will do so for the square orders \( p = 4, 9, 25 \). Along the way, we will see how this is tied up to continued fractions, Platonic solids, and \( \pi \) formulas.

**II. The \( k \)-function of Square Order**

In Part I, we gave some properties of what we called the \( k \)-function, generalizations of Ramanujan’s eta quotient \( \lambda_n \) and the Weber functions. We found that there were certain identities they obeyed, analogous to the ones for the Weber functions.

For the \( k \)-function of square and of course, composite order \( p = 4, 9, 25 \), such identities were hard to find. However, for what identities they may seemingly lack, they make up for it in the elegance and simplicity of their modular relations. First, we will tabulate the basic facts about these functions,

<table>
<thead>
<tr>
<th>( \Gamma_0(p) )</th>
<th>Degree</th>
<th>Modular Relation</th>
<th>McKay/Thompson #</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_0(4) )</td>
<td>( 2^2+2 )</td>
<td>( P(y)_4 )</td>
<td>4C</td>
</tr>
<tr>
<td>( \Gamma_0(9) )</td>
<td>( 3^2+3 )</td>
<td>( P(y)_9 )</td>
<td>9B</td>
</tr>
<tr>
<td>( \Gamma_0(25) )</td>
<td>( 5^2+5 )</td>
<td>( P(y)_{25} )</td>
<td>25B</td>
</tr>
</tbody>
</table>

where the polynomial \( P(y) \) will be given below. There are certain differences when dealing with square orders. One, for prime \( p \), the degree of the modular relation is just \( p+1 \) but for composite order \( m^2 \), the degree is \( m^2+m \). Another, for the former, there were two ways that the \( j \)-function with argument \( \tau \) was expressed in terms of eta quotients with argument involving only \( \tau \) and \( p \), namely \( \frac{\eta(\tau / p)}{\eta(\tau)} \) and \( \frac{\eta(p\tau)}{\eta(\tau)} \). For the latter, there is a third way, given by \( \frac{\eta(\tau / m)}{\eta(m\tau)} \) though strictly speaking this no longer falls under our definition of the \( k \)-function.

As with Part I, the author took the liberty of doing a small transformation on the relations in Chen and Yui’s table which were originally in the variable Hauptmodul \( f \), given by,

\[ f = \frac{p^{r/2}}{x} \]
with \( r \) still as the *modular ratio*. (The negative sign in Part I was inserted to accommodate the Weber functions.) With a further transformation involving a *translation*, we get much simplified and elegant formulas for the \( j \)-function. If we let the equation be \( j(\tau) = P(y) \), then the general form of the solution is given by,

\[
y = c' + r
\]

where \( c \) is an eta quotient, and \( r \) the modular ratio.

**A. Order 4**

*Form 1*,

\[
 j(\tau) = \frac{(y^2 - 48)^3}{y^2 - 64}
\]

\[
y_1 = \left( \frac{\eta(\tau/4)}{\eta(\tau)} \right)^8 + 8 \quad y_2 = \left( \frac{2\eta(4\tau)}{\eta(\tau)} \right)^8 + 8
\]

*Form 2*,

\[
 j(\tau) = \frac{(z^2 + 192)^3}{(z^2 - 64)^2}
\]

\[
z = \left( \frac{\eta(\tau/2)}{\eta(2\tau)} \right)^8 + 8
\]

Given the \( k \)-functions \( f_4 \) of order 4,

\[
f_{40+q} = e^{\pi \eta q/4} \left( \frac{1}{4} (\tau + 3q) \right)^{\eta(\tau)}
\]

\[
f_{44} = \frac{2\eta(4\tau)}{\eta(\tau)}
\]

for \( q = 0 \) to 3. Then \( f_4^8 + 8 \) are *5 of the 6 roots* of the form 1 identity.

Since this is an even order, notice that its form is slightly different from the others. And for the composite orders the \( k \)-function will not cover all the roots. However, if the modular relation is given in the form,

\[
j(\tau) = \frac{(v - 48)^3}{v - 64}
\]
where \( v = (f_4^3 + 8)^3 \), then there are in fact just three roots and our five \( k \)-functions \( f_4 \) can express them all, with repetition. A similar situation happens for order 9. For form 2, by doing the transformation,

\[
z = -16w + 8
\]

we get,

\[
j(\tau) = \frac{4^3(w^2 - w + 1)^3}{w^2(w - 1)^2}
\]

where \( w \) is satisfied by the elliptic lambda function, \( \lambda(t) \), and all six roots can be expressed as eta quotients as was discussed in Part 1.

**B. Order 9**

**Form 1,**

\[
j(\tau) = \frac{y^3(y^3 - 24)^3}{y^3 - 27}
\]

\[
y_1 = \left( \frac{\eta(\tau/9)}{\eta(\tau)} \right)^3 + 3 \quad y_2 = \left( \frac{3\eta(9\tau)}{\eta(\tau)} \right)^3 + 3
\]

**Form 2,**

\[
j(\tau) = \frac{z^3(z^3 + 216)^3}{(z^3 - 27)^3}
\]

\[
z = \left( \frac{\eta(\tau/3)}{\eta(3\tau)} \right)^3 + 3
\]

Given the \( k \)-functions \( f_9 \) of order 9,

\[
f_{90q} = e^{\pi i q/3} \left( \frac{1}{9} (\tau + 4q) \right) \eta(\tau)
\]

\[
f_{99} = \frac{3\eta(9\tau)}{\eta(\tau)}
\]

for \( q = 0 \) to 8. Then \( f_9^3 + 3 \) are 10 of the 12 roots of form 1 (of order 9).
C. Order 25

Form 1,

\[ j(\tau) = -\frac{(s^{20} + 12s^{15} + 14s^{10} - 12s^5 + 1)^3}{s^{25}(s^{10} + 11s^5 - 1)} \]

\[ s^{-1} - s = \frac{\eta(\tau/25)}{\eta(\tau)} + 1 \]

\[ s^{-1} - s = \frac{5\eta(25\tau)}{\eta(\tau)} + 1 \]

Form 2,

\[ j(\tau) = -\frac{(r^{20} - 228r^{15} + 494r^{10} + 228r^5 + 1)^3}{r^5(r^{10} + 11r^5 - 1)^3} \]

\[ r^{-1} - r = \frac{\eta(\tau/5)}{\eta(5\tau)} + 1 \]

Given the \( k \)-functions \( f_{25} \) of order 25,

\[ f_{250,q} = e^{\pi i q} \frac{\eta \left( \frac{1}{25}(\tau + 12q) \right)}{\eta(\tau)} \]

\[ f_{275} = \frac{5\eta(25\tau)}{\eta(\tau)} \]

for \( q = 0 \) to 24. Let \( s^{-1} - s = f_{25} + 1 \), then all the \( s \) are 52 of the 60 roots of form 1 (of order 25).

For order 25, we have a different situation. Given our preliminary modular relation of degree 30, after we do the translation it does not give us a form 1 identity with powers only in multiples of 5. But if we do still another transformation, then we get the elegant form above.

III. Continued Fractions And Platonic Solids

For the first kind of continued fraction given in the Introduction, the general form is given by the Rogers-Ramanujan continued fraction, \( r(\tau) \),

\[ r(\tau) = \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \ldots}}}} \]
where \( q = e^{2\pi \tau} = \exp(2\pi i \tau) \). How do we find the evaluation of \( r(\tau) \)? One answer is to find a modular relation between \( j(\tau) \) and \( r(\tau) \), and which in fact has been given earlier, namely,

\[
j(\tau) = \frac{(r^{20} - 228r^{15} + 494r^{10} + 228r^5 + 1)^3}{r^5(r^{10} + 11r^5 - 1)^3}
\]

which is also known as the icosahedral equation (see Duke’s paper). Thus, if one has \( j(\tau) \), then it is a simple matter of substitution or getting resultants by eliminating \( j \) between the above polynomial and the Hilbert class polynomial and we can solve for \( r(\tau) \).

If expressed as eta quotients, then \( r(\tau) \) is given by the quadratic relation,

\[
r^{-1} - r = \frac{\eta(\tau/5)}{\eta(5\tau)} + 1
\]

Examples:

Given \( d = 4(1) \), then \( \tau = \sqrt{-1} = i \) and \( j(i) = 12^3 \). Substituting this value into the polynomial above and factoring, we get several factors. To determine which of the factors is the correct one, we can numerically solve the quadratic relation above. We find that it is then the root of a quartic, and this root is explicitly given by,

\[
r(i) = \sqrt{\phi + 2} - \phi
\]

For the second example, we used \( d = 4(4) \), \( \tau = \sqrt{-4} = 2i \) and \( j(2i) = 66^3 \). Using the same method above, we get,

\[
r(2i) = x
\]

where \( x \) is the appropriate root of,

\[
x^{-1} - x = 1 + 5^{3/4} \phi + 5^{1/2} \phi^2
\]

and so on for all the values of the \( j \)-function.

For the second kind of continued fraction, the general form is given by,

\[
u(\tau) = \frac{\sqrt{2} q^{1/8}}{1 + q^{1/8}} \frac{q}{1 + q + \frac{q^2}{1 + q^2 + \frac{q^3}{1 + q^3 + \ldots}}}
\]
still with \( q = e^{2\pi \tau} = \exp(2\pi i \tau) \). The modular relation between \( j(t) \) and \( u(t) \) has also been given earlier, though in a different guise. Given the original \( j \)-function formula in Chen and Yui’s table involving \( ?_0(4) \),

\[
j(\tau) = \frac{(f^2 + 256f + 4096)^3}{(f + 16)f^4}
\]

where \( f \) is a Hauptmodul, we can transform that into the form 1 of order 4 given earlier. However, we can also transform it differently. Let \( f = -16\tau \), then,

\[
j(\tau) = \frac{16(x^2 - 16x + 16)^3}{x^4(1 - x)} \quad \text{(eq.3.1)}
\]

This has the six solutions,

\[
x_1 = \left( \frac{\eta^4(\tau) \eta^2(4\tau)}{\eta^6(2\tau)} \right)^4
\]

\[
x_2 = -\left( \frac{\eta(\tau)}{\sqrt{2}\eta(4\tau)} \right)^8
\]

\[
x_3 = \left( \frac{\eta^4(\beta) \eta^2(4\beta)}{\eta^6(2\beta)} \right)^4
\]

where \( \beta = -\frac{1}{\tau + a} \) for \( a = 0, 1, 2, 3 \).

Now let \( x = 1 - y \) and we have the variant of the above as,

\[
j(\tau) = \frac{16(y^2 + 14y + 1)^3}{(y - 1)^4y} \quad \text{(eq.3.2)}
\]

with solutions,

\[
y_1 = \left( \frac{2\eta^2(\tau) \eta^4(4\tau)}{\eta^6(2\tau)} \right)^4
\]

\[
y_2 = 1 + \left( \frac{\eta(\tau)}{\sqrt{2}\eta(4\tau)} \right)^8
\]
\[ y_3 = \left( \frac{2\eta^2(\beta)\eta^4(4\beta)}{\eta^6(2\beta)} \right)^4 \]

where \( \beta = -\frac{1}{\tau + a} \) for \( a = 0, 1, 2, 3 \).

While we can express the roots of eq.3.2 as \( y = 1 - x \) (which indeed is the case for \( y_2 \)), notice that we can also express them as slightly different eta quotients. Why do we go through these contortions? It is because we are after \( y_1 \), since, and here is the surprise,

\[ u^2(\tau) = w = y_1^{1/4} = \frac{2\eta^2(\tau)\eta^4(4\tau)}{\eta^6(2\tau)} \]

where \( w \) is an indefinite variable and thus,

\[ j(\tau) = \frac{16(w^8 + 14w^4 + 1)^3}{w^4(w^4 - 1)^4} \]

In analogy to the icosahedron, this can be called the octahedral equation.

Examples:

We will use the same \( d \) earlier, as these have simple \( j(t) \). Substituting \( j(i) = 12^3 \) into the above equation, it factors and using the eta quotient formula to determine the correct one, we find the root,

\[ w = -1 + \sqrt{2} \]

since \( u^2(\tau) = w \), then,

\[ u(i) = \sqrt{-1 + \sqrt{2}} \]

Doing the same for \( j(2i) = 66^3 \), we get, \[ u(2i) = \sqrt{w} = 0.293985 \ldots \]

where \( w \) is the appropriate root of,

\[ w + w^{-1} = 2(1 + \sqrt{2})^2 \]

and so on.

Now what is the connection to Platonic solids? To recall, the Platonic solids, or regular polyhedra, are polyhedra with equivalent faces composed of convex regular polygons. There are
only five such solids, namely the tetrahedron, cube, octahedron, dodecahedron, and icosahedron, with 4, 6, 8, 12, 20 faces, respectively.

However, for every polyhedron there is a dual, that is, another polyhedron in which faces and vertices have complementary positions. The dual of a Platonic solid is a Platonic solid. For the tetrahedron, it is self-dual; for the octahedron, it is the cube; and for the icosahedron, it is the dodecahedron. Thus, of the symmetry groups of the Platonic solids, there are just three polyhedral groups: the tetrahedral group of order 12, the octahedral group of order 24, and the icosahedral group of order 60. We shall see that there is a continued fraction that can be associated with the last two groups (and perhaps also to the first).

To see the connection between the first kind of continued fraction and the Platonic solids, we have to look at the icosahedron’s projective geometry. Given an icosahedron with unit circumradius, by projecting its vertices using a stereographic projection from the south pole of its circumsphere onto the plane and expressing these vertex locations as roots of an algebraic equation, we get,

\[ P(z, 1) = z(z^{10} + 11z^5 - 1) = 0 \]

If an icosahedron with unit inradius is projected, then the equation expressing the face center positions is given by,

\[ H(z, 1) = z^{20} - 228z^{15} + 494z^{10} + 228z^5 + 1 \]

Do the coefficients of the equations look familiar? They are the same as those in the denominator and numerator of our icosahedral equation!

\[ j(\tau) = -\frac{(r^{20} - 228r^{15} + 494r^{10} + 228r^5 + 1)^3}{r^5(r^{10} + 11r^5 - 1)^5} \]

Note that this has exactly 60 roots and the icosahedral group is of order 60.

For the second kind of continued fraction, we have to look at the octahedron. Given an octahedron with unit circumradius, if we project it likewise, the vertex locations are given by,

\[ P(z, 1) = z(z^4 - 1) = 0 \]

and with unit inradius, the projection gives the face center positions as,

\[ H(z, 1) = z^8 + 14z^4 + 1 \]

Compare to the octahedral equation,

\[ j(\tau) = \frac{16(w^8 + 14w^4 + 1)^3}{w^4(w^4 - 1)^4} \]

which, perhaps not surprisingly, has 24 roots and the octahedral group is of order 24.
For the last group, the tetrahedral group, we can also come up with equations for it analogous to the ones above. Given a tetrahedron with unit circumradius, if we project it in the same manner, then the vertex locations are given by,

\[ P(z, l) = z^4 - 2\sqrt{2}z = 0 \]

and with unit inradius, the face center positions are,

\[ H(z, l) = 2\sqrt{2}z^3 + 1 \]

For the previous two polyhedral groups, we have used \( j \)-function formulas from the square orders \( p = 4, 25 \). But there’s still the square order \( p = 9 \). Given the form 2 formula,

\[ j(\tau) = \frac{z^3(z^3 + 216)^3}{(z^3 - 27)^3} \]

if we transform that by letting \( z = -3\sqrt{2}x \), then we get,

\[ j(\tau) = -\frac{1728(x^4 - 2\sqrt{2}x)^3}{(2\sqrt{2}x^3 + 1)^3} \]

Compare coefficients to the equations for the tetrahedron. The formula has exactly 12 roots and the tetrahedron is of order 12.

I am not aware of any continued fraction associated with the symmetry group of the tetrahedron though I assume there must be one. Since each of the square orders (for \( p = 4, 9, 25 \)) have a \( j \)-function formula that can be associated with a polyhedral group, I find it exceedingly interesting that there are only three kinds for both order and group, considering that the reason for the former is only basic arithmetic.

For a more detailed discussion on the relation of Ramanujan’s continued fractions to the icosahedral and octahedral groups, the reader is referred to William Duke’s excellent paper “Continued Fractions and Modular Functions”. For more on the polyhedral equations used in this paper, see [http://mathworld.wolfram.com/IcosahedralEquation.html](http://mathworld.wolfram.com/IcosahedralEquation.html) and the links within it.

### VI. Pi Formulas

Before we go to the pi formulas, we can discuss some curious numerical phenomena. Given even discriminants \( d = 4m \) with class number 2, \( m = 13, 37 \), why is it that,

\[ e^{\pi \sqrt{3}} = (12\sqrt{2})^4 + 103.95... \]
\[ e^{\pi \sqrt{57}} = (84\sqrt{2})^4 + 103.999978... \]

Trying to recover the first few terms of its series expansion, we get \{1, 104, 4372, 96256…\} (unsigned) and OEIS gives this as A007267, the McKay-Thompson class 2A of Monster.
Given odd discriminants $d = 3m$ with class number 2 ($m = 17, 41, 89$), we have,

\[
\begin{align*}
  e^{x \sqrt{3/17}} &= 12^3 + 41.56... \\
  e^{x \sqrt{3/41}} &= 48^3 + 41.993... \\
  e^{x \sqrt{3/89}} &= 300^3 + 41.99997...
\end{align*}
\]

and the first few terms of its series expansion \{1, 42, 783, 8672…\} (unsigned) is A030197, the McKay-Thompson class 3A of Monster.

For $d = 7m, m = 13, 61$, we have,

\[
\begin{align*}
  e^{x \sqrt{7/13}} &= (4^3 - 1) + 9.33... \\
  e^{x \sqrt{7/61}} &= (22^3 - 1) + 9.995...
\end{align*}
\]

with \{1, 10, 51, 204, 681…\} (unsigned) which OEIS says is A030183, the McKay-Thompson class 7A of Monster.

To recall, in Part I, we had similar approximations that involved class $pB$ of Monster. And now we have class $pA$. Thus, there should be another set of modular relations. Conveniently for us, Chen and Yui have already tabulated this in the same paper (p. 285) from where we based our Table 1. Actually, even without this second table, we can construct it ourselves since the two sets of modular relations are related, i.e. we can transform the first set into the second. The transformation is given by:

First transformation,

\[
  t_p = (-1)^{p-1} \left( \frac{x^2 + p^{r/2}}{x} \right)
\]

where $r$ is still the modular ratio. By eliminating $x$ between the relations in our Table 1 and this equation, we get a quadratic modular relation between $j(\tau)$ and $t$ for our Table 2. However, for this section, we will again limit ourselves to the prime orders $p = 2, 3, 5, 7, 13$.

<table>
<thead>
<tr>
<th>$\Gamma$</th>
<th>Modular Relation</th>
<th>McKay/Thompson #</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_0(2)^+$</td>
<td>$j^2 - (t^2 + 49t - 6656)j + (t + 272)^3 = 0$</td>
<td>2A</td>
</tr>
<tr>
<td>$\Gamma_0(3)^+$</td>
<td>$j^2 - (t^2 - 18t - 944)(t + 54)j + (t + 54)(t + 246)^3 = 0$</td>
<td>3A</td>
</tr>
<tr>
<td>$\Gamma_0(5)^+$</td>
<td>$j^2 - (t^5 + 30t^4 - 310t^3 - 13700t^2 - 38424t + 614000)(t + 260)(t + 6380)^3 = 0$</td>
<td>5A</td>
</tr>
<tr>
<td>$\Gamma_0(7)^+$</td>
<td>$j^2 - (t^4 + 10t^3 - 170t^2 - 1710t - 371)(t^2 + 5t - 96)(t + 13)j + (t + 13)^2 (t^2 + 250t + 3529)^3 = 0$</td>
<td>7A</td>
</tr>
<tr>
<td>$\Gamma_0(13)^+$</td>
<td>(and so on)</td>
<td>13A</td>
</tr>
</tbody>
</table>
where the equation for 13A is rather tedious to write down. Unlike our Table 1 which was a modification (to give us smaller coefficients, among other reasons), this Table 2 is identical to Chen and Yui’s, though the expressions have been simplified.

To see the connection to the approximations given earlier as well as to pi formulas, we need to do a second transformation, though we will do so only for the prime $p$ such that the modular ratio $r$ is divisible by 4, namely for $p = 2, 3, 7$.

**Second transformation**, 

$$t_p = h - 2p^{r/4}$$

By substituting this into the appropriate equations for $p = 2, 3, 7$, we get,

$$j^2 - (h^2 - 207h + 3456) j + (h + 144)^3 = 0$$

$$j^2 - h(h^2 - 126h + 2944) j + h(h + 192)^3 = 0$$

$$j^2 - h(h^2 - 21h + 8)(h^4 - 42h^3 + 454h^2 - 1008h - 1280) j + h^2(h^2 + 224h + 448)^3 = 0$$

with transformations $t_2 = h - 128$, $t_3 = h - 54$, $t_7 = h - 13$, respectively. (Technically, for the third, the transformation should be $t_7 = h - 14$ but we chose the other one for reasons which will be apparent later.)

The relationship between the two sets of modular relations, namely, the linear ones in Table 1 ($j$ and $x$) and the quadratic ones in these three equations ($j$ and $h$), can easily be ascertained by eliminating $j$ between them. They are given below in terms of the Weber and $k$-functions $f_p(\tau)$.

For our examples the convention used will be the same as in part I,

a) $f(d)$ for odd discriminants $d$ implying $\tau$ is of form $\tau = (1 + \sqrt{-d})/2$.

b) $f(4m)$ for even discriminants $d = 4m$ implying $\tau = \sqrt{-m}$.

**A. Order 2**

Let,

$$j^2 - (h^2 - 207h + 3456) j + (h + 144)^3 = 0$$

(eq. 1a)

then as a cubic in $h$, its roots are given by,

$$h_2(\tau) = -\frac{(f_i^{24} - 64)^2}{f_i^{24}}$$

(eq. 1b)
where the $f_i$ are the three Weber functions and the appropriate $h_2(\tau)$ is a root of a Ramanujan class polynomial.

Example 1

Let $d = 148 = 4 \times 37$. Since,

$$j(4 \times 37) = 60^3(2837 + 468\sqrt{37})^3$$

substituting this value for $j(\tau)$ into the equation above (eq.1a), we find that it factors, one of which surprisingly is integral, given by,

$$h = -(84\sqrt{2})^4$$

If we use the Weber function $f = e^{-\pi/24} \frac{\eta((\tau + 1)/2)}{\eta(\tau)}$, and the equation between $h_2$ and $f_i$ (eq.1b), we get,

$$h_{20}(4 \times 37) = -(84\sqrt{2})^4$$

and which explains the approximation,

$$e^{\pi \sqrt{37}} = (84\sqrt{2})^4 + 103.999978...$$

as well as the denominator of the pi formula below (and its negative sign explains why it is an alternating sum),

$$\frac{1}{4\pi} = \sum_{n=0}^{\infty} s_1(-1)^n \frac{21460n + 1123}{(84\sqrt{2})^4n^2}$$

where $s_1 = \frac{(4n)!}{n!^4}$.

In general, it explains all the denominators of the formulas in Pi Formulas, Ramanujan, and the Baby Monster Group (in the section Ramanujan Class Polynomials, by this author) which uses even discriminants $d = 4m$ for odd $m$. In that paper, we used the Ramanujan g- and G-functions, though since squaring was involved, it had the disadvantage of losing the negative sign which is retained in using either (eq.1a) or (eq.1b), equations which were hinted at in that previous paper. For more details in finding pi formulas of the same kind, the reader is referred to that paper.

Example 2

Let $d = 232 = 4 \times 58$. Since,

$$j(4 \times 58) = 30^3(140989 + 26163\sqrt{29})^3$$
then (eq.1a) has the root,

\[ h = 396^4 \]

Using the Weber function \( f_1 = \frac{\eta(\tau / 2)}{\eta(\tau)} \) and (eq.1b), we find that,

\[ h_{21}(4 \ast 58) = 396^4 \]

compare to the denominator of the pi formula,

\[
\frac{1}{16\pi} = 2\sqrt{2} \sum_{n=0}^{\infty} \frac{26390n + 1103}{396^{4n+2}}
\]

This explains the denominators of the pi formulas which this time are just plain infinite sums (as well as the corresponding approximations) in the same paper and section which uses \( d = 4m \), this time for \textbf{even} \( m \).

\textbf{B. Order 3}

Let,

\[ j^2 - h(h^2 - 126h + 2944)j + h(h + 192)^3 = 0 \quad \text{(eq.2a)} \]

then as a quartic in \( h \), its roots are given by,

\[ h_3 = \frac{(f_3^{12} + 27)^2}{f_3^{12}} \quad \text{(eq.2b)} \]

where the \( f_3 \) are the four \( k \)-functions of order 3,

\text{Example 1}

\text{Let } d = 24 = 4*6. \text{ We have,}

\[ j(4 \ast 6) = 12^3(1 + \sqrt{2})^2(5 + 2\sqrt{2})^3 \]

and (eq.2a) has the root,

\[ h = 6^3 \]

Using \( f_{30} = \frac{\eta(\tau / 3)}{\eta(\tau)} \) and (eq.2b) we find that,

\[ h_{30}(4 \ast 6) = 6^3 \]
and we have the pi formula,

\[ \frac{1}{\pi} = \frac{1}{3\sqrt{3}} \sum_{n=0}^{\infty} s_2 \frac{6n+1}{6^{3n}} \]

where \( s_2 = \frac{2^{2n} 3^{3n} (1/2)_n (1/3)_n (2/3)_n}{n!^3} \) and \((a)_n\) is the rising factorial or Pochhammer symbol.

In general, this explains the formulas in the paper *Ramanujan’s Other Pi Formulas* which uses even \( d = 4m \), regardless of the nature of \( m \).

Example 2

Let \( d = 267 = 3 \times 89 \). Since,

\[ j(267) = -240^3 (500 + 53\sqrt{89})^2 (625 + 53\sqrt{89})^3 \]

(eq.2a) has the root,

\[ h = -300^3 \]

Using \( f_{31} = e^{\pi i/12} \frac{\eta((\tau + 1)/3)}{\eta(\tau)} \) and (eq.2b) we get,

\[ h_{31}(267) = -300^3 \]

and which explains,

\[ e^{\pi \sqrt{89/3}} = 300^3 + 41.99997... \]

Compare to the denominator of the pi formula which is also an alternating sum,

\[ \frac{1}{\pi} = \frac{1}{1500\sqrt{3}} \sum_{n=0}^{\infty} (-1)^n \frac{14151n + 827}{300^{3n}} \]

and the new pi formulas found by the author which uses odd \( d \). In that paper, we used a similar approach, namely through certain Dedekind eta quotients, though to determine its defining polynomial we had to use Integer Relations. With eq.2a (which was promised in that paper), if one has \( j(\tau) \), then it reduces to simple polynomial factoring.

C. Order 7

Let,

\[ j^2 - h(h^2 - 21h + 8)(42h^3 + 454h^2 - 1008h - 1280) j + h^2(h^2 + 224h + 448)^3 = 0 \]
(eq.3a). As an octic in $h$, its roots are,

$$h_7 = \frac{(f_7^4 + 7)^2}{f_7^4} - 1$$  \hspace{1cm} (eq.3b)

where the $f_i$ are the eight $k$-functions of order 7.

Example 1

Let $d = 532 = 4 \times 133$. (This has class number 4 and since its quartic Hilbert class polynomial is unwieldy, we will not write it down.) Equation (eq.3a) has the root,

$$h = (13 + 7\sqrt{7})^3$$

Using $f_{70} = \frac{n(\tau / 7)}{n(\tau)}$ and (eq.3b) we have,

$$h_{70}(4 \times 133) = (13 + 7\sqrt{7})^3$$

Example 2

Let $d = 427 = 7 \times 61$. Since,

$$j(427) = -5280^3(236674 + 30303\sqrt{61})^3$$

(eq.3a) has a root,

$$h = -22^3$$

Using $f_{71} = e^{\pi i/4} \frac{((\tau + 3)/7)}{\eta(\tau)}$ and (eq.3b) we find that,

$$h_{71}(427) = -22^3$$

thus,

$$e^{\frac{\pi}{6\sqrt{61}\sqrt{7}}} = (22^3 - 1) + 9.995...$$

Some remarks for this section: First, the reason why we used the transformation $t_7 = h - 13$ is that for certain $d$ (usually $d = 7m$ though with exceptions) the relevant root of the transformed equation is a perfect cube. For example, for $d = 7m$ with class number 4, $m = 29, 37, 109$, we find that,

$$h_{71}(203) = -(3 + \sqrt{29})^3$$

$$h_{71}(259) = -(5 + \sqrt{37})^3$$
Second, the important functions, at least for this section, are the Weber functions $f, f_1$ and the $k$-functions $f_{30}, f_{31}, f_{70},$ and $f_{71}$. In general, $f_{30}$ and $f_{70}$ for even discriminants ($f_1$ for $d = 4m, m$ even) and $f_{31}$ and $f_{71}$ for odd discriminants ($f$ for $d = 4m, m$ odd).

Third, the approximations are of the form,

$$e^{x \sqrt{d/p}} \approx |h_p| + |v_p|$$

for appropriate $h_p$ and where the “excess” approaches $v_p = 104, 42, 10-1$ for $p = 2, 3, 7$, respectively, as $d \to \infty$.

Fourth, it is very tempting to speculate that since $h_2$ and $h_4$ are involved in pi formulas, then perhaps $h_7$ should also be, though as far as I know, there is no Ramanujan-type pi formula that involves the product of rising factorials $(1/2)^n (1/7)^n (6/7)^n$.

V. Conclusion

We have finally accomplished what we have set out to do, namely to provide a generalization of Ramanujan’s eta quotient $\lambda_n$ (which was of order $p = 3$) and to find analogous functions for all appropriate orders, which were for $p = 2, 3, 4, 5, 7, 9, 13, 25$. Interestingly, the existence for these orders of explicit modular relations with rational coefficients involving the $j$-function and a Hauptmodul depended on basic arithmetical properties of the integer $24$, that is, they were the only $p$ such that $p-1$ evenly divides into $24$.

These functions also obeyed certain identities among themselves that were analogues of Weber’s and Jacobi’s identities. Their functional equations were similar to those for the $j$-function, though if seen in the context of a pair of complementary $k$-functions.

We have discussed eight formulas for the $j$-function plus an additional three that were variants from the square orders, formulas of varying degrees of simplicity and aesthetic form. We also saw that the three square orders had a connection to continued fractions, the Platonic solids, and to the three polyhedral groups.

Before we conclude the paper, one question comes to mind. Does a dissection of Ramanujan’s seemingly magical work, whether on continued fractions, pi formulas, eta quotients and others, give the impression (to quote John Keats’ lament on Newtonian physics) of “unweaving a rainbow”, now that we partially understand what is going on? I think not. As Richard Dawkins points out in his book, there is a kind of poetry and aesthetics in science (or in our case, mathematics). The fact still remains that Ramanujan’s “rainbow” is there to begin with, he was among the first to see it and point out where it is, and now that we know where to look, if we follow it to its end, it really does lead to a pot of gold.

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References:

10. et al.