Ramanujan’s Cubic Continued Fraction (And Others)

By Titus Piezas III

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Dedicated to the memory of Isaac Asimov (1920-1992).

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I. Introduction

In “The k -Function: Continued Fractions and Platonic Solids (Part II)” [1] we discussed two of Ramanujan’s continued fractions, namely the quintic Rogers-Ramanujan $R(q)$ and an octic continued fraction $U(q)$. We found that there was an intriguing relationship between these fractions, Dedekind eta quotients, the $k$-function of square order, and Platonic solids.

Of the three polyhedral groups, namely the tetrahedral, octahedral, and icosahedral groups, we saw that for $R(q)$, it was connected to the symmetry group of the icosahedron. For $U(q)$, it was for the octahedron. However, for the tetrahedron, we didn’t find then a continued fraction related to its symmetry group. In this paper, we will discuss Ramanujan’s cubic continued fraction $C(q)$ which might turn out to be the one related to it. First though we can take a second look at $R(q)$ and $U(q)$.

The Rogers-Ramanujan continued fraction, $R(q)$ is given as,

$$R(q) = \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \ldots}}}}$$

If we set $q = e^{2\pi i \tau} = \exp(2\pi i \tau)$, then for this particular subset, call it $r(\tau)$, an explicit modular relation $M$ can be found between it and the j-function, $j(\tau)$, and which can enable us to find its exact evaluation. In general, for any continued fraction we will be looking for a relation $M$ defined by two features: 1st, it must have rational coefficients and, 2nd, it must be linear in $j(\tau)$. For $r(\tau)$, this is given by, let $r(\tau) = r$, 
\[ j(\tau) = -\frac{(r^{20} - 228r^{15} + 494r^{10} + 228r^5 + 1)^3}{r^5(r^{10} + 11r^5 - 1)^5} \]

which is also known as the *icosahedral equation*. Expressed as Dedekind eta quotients, \( r(\tau) \) is given by the quadratic relation,

\[ r^{-1} - r = \frac{\eta(\tau / 5)}{\eta(5\tau)} + 1 \]

(Later we will see that Ramanujan’s cubic continued fraction will involve a *cubic* relation to certain eta quotients.) As an example, we can give Ramanujan’s own example that he gave in his first letter to Hardy,

\[ r(\sqrt{-1}) = \sqrt{\phi + 2} - \phi = \frac{e^{-2\pi/5}}{e^{-2\pi}} = 0.2840... \]

\[ 1 + \frac{e^{-4\pi}}{1 + \frac{e^{-6\pi}}{1 + \frac{e^{-6\pi}}{1 + \ldots}}} \]

where \( \phi = (1 + \sqrt{5})/2 \) is the *golden ratio*. It uses the argument \( \tau = \sqrt{-1} \) of the (negated) discriminant \( d = 4(1) \), with class number 1 thus giving a value \( j(\tau) = 12^3 \). Substituting this value into the icosahedral equation, one of its factors will be given by \( r(\sqrt{-1}) \).

If we define our \( q \) differently, say \( q = \exp(2\pi i \tau / 5) \), we can find a variant, call it \( s(\tau) \), which is a root of a different \( j \)-function formula,

\[ j(\tau) = -\frac{(s^{20} + 12s^{15} + 14s^{10} - 12s^5 + 1)^3}{s^{25}(s^{10} + 11s^5 - 1)} \]

or as eta quotients,

\[ s^{-1} - s = \frac{\eta(\tau / 25)}{\eta(\tau)} + 1 \]

**Example:**

By substituting the same \( j(\tau) = 12^3 \) into the equation above, one of its factors is given by the value of the continued fraction,

\[ s(\sqrt{-1}) = (3\sqrt{\phi + 2} - 4 - \phi)^{1/5} = \frac{e^{-2\pi / 25}}{e^{-2\pi/5}} = 0.6154... \]

\[ 1 + \frac{e^{-4\pi / 5}}{1 + \frac{e^{-6\pi / 5}}{1 + \ldots}} \]
If we define our $q$ a third way, say $q = \exp(10\pi \tau)$, we can find still another variant, call it $t(\tau)$, which is a root of a third $j$-function formula. Its actual formula in the variable $t$ is unwieldy, but with a change of variable to $x$, we have the simpler and familiar form,

$$j(\tau) = -\frac{(x^{20} + 12x^{15} + 14x^{10} - 12x^5 + 1)^3}{x^{25}(x^{10} + 11x^5 - 1)}$$

where $x$ is related to the original variable $t$ by,

$$(t^2 + t - 1)x^2 + (t^2 - 4t - 1)x - (t^2 + t - 1) = 0$$

This factors over $\sqrt{5}$, and we have the two solutions for $t$ though only one is the appropriate value,

$$t_1 = \frac{1 - \phi x}{\phi + x}, \quad t_2 = \frac{\phi + x}{\phi x - 1}$$

where $\phi$ is still the golden ratio. Or if expressed as eta quotients,

$$t^{-1} - t = \frac{\eta(\tau)}{\eta(25\tau)} + 1$$

While the linear relation between $x$ and $t$ has a square root, plugging it into the formula will yield an equation with only rational coefficients.

Example:

Substituting the same $j(\tau) = 12^3$ into the formula, we find that,

$$t(\sqrt{-1}) = \frac{1 - \phi x}{\phi + x} = \frac{e^{-2\pi}}{1 + \frac{e^{-10\pi}}{1 + \frac{e^{-20\pi}}{1 + \frac{e^{-30\pi}}{1 + \ldots\ldots}}}} = 0.001867 \ldots$$

where $x = (3\sqrt{\phi + 2 - 4 - \phi})^{1/5}$.

Thus for a single value of the $j$-function, by substituting it into the three formulas and solving for $r$, $s$, and $t$, we can automatically find the evaluation of the Rogers-Ramanujan continued fraction for three arguments $q$, either as the exact algebraic expression or its defining polynomial.

We can then ask the simple-looking question: For how many arguments $q$ can we do this? In other words, how many distinct $M$ (as we defined it) can we find? It turns out the answer may lie in the connection of the continued fraction to the modular group $\Gamma(n)$. Before we elaborate
on this, we can discuss the related phenomenon of \textit{j-function argument n-tuplication} illustrated by the next continued fraction.

The octic continued fraction $U(q)$ is given by,

$$U(q) = \frac{\sqrt{2} q^{1/8}}{1 + \frac{q}{1 + q + \frac{q^2}{1 + q^2 + \frac{q^3}{1 + q^3 + \ldots}}}}$$

Let $q = e^{2\pi i z} = \exp(2\pi i z)$, with complex number $z$ in lieu of $\tau$, and we have the particular subset $u(z)$ which can be expressed as eta quotients,

$$u(z) = \frac{\sqrt{2} \eta(z) \eta^2(4z)}{\eta^3(2z)}$$

This continued fraction is connected to the modular group $\Gamma(4)$ so perhaps it is not surprising that it satisfies a number of beautiful $j$-function $n$-tuplication formulas for $n$ a positive or negative power of 2, given by,

$$j(z) = \frac{16(u^{16} + 14u^8 + 1)^3}{u^8(u^8 - 1)^4}$$

$$j(2z) = \frac{256(u^{16} - u^8 + 1)^3}{u^{16}(u^8 - 1)^2}$$

$$j(4z) = -\frac{16(u^{16} - 16u^8 + 16)^3}{u^{32}(u^8 - 1)}$$

$$j(z/2) = \frac{4(u^{16} + 60u^{12} + 134u^8 + 60u^4 + 1)^3}{u^4(u^4 + 1)^2(u^4 - 1)^8}$$

$$j(z/4) = \frac{16(x^8 + 28x^6 - 10x^4 - 4x^2 + 1)^3}{x^4(x^2 - 1)^8(2x^2 - 1)}$$

where $x = u + u^{-1}$.

There is also one for $n = 8$, though one has to use square root of a variable already,

$$j(8z) = \frac{4(y^4 + 60y^3 + 134y^2 + 60y + 1)^3}{y(y + 1)^2(y - 1)^8}$$

where $y = \sqrt{1 - u^8}$.

Example:

Still using the argument $z = \sqrt{-1}$, from the previous paper we know that,
Substituting this single value of the continued fraction into the formulas above, we find the following values of the j-function at various arguments,

\[ j(z) = 12^3, \quad j(2z) = 66^3, \quad j(4z) = 3^3(724 + 513\sqrt{2})^3, \]

\[ j(z/2) = 66^3, \quad j(z/4) = 3^3(724 + 513\sqrt{2})^3 \]

and where \( j(8z) \) is the root of a quartic with rather large coefficients. (The fact that some values are identical is just a peculiarity of the particular \( \tau \) we have chosen.)

Another way to look at these formulas is that by setting \( z = \tau / n \) they can also be considered as different continued fractions at the arguments \( q = \exp(2\pi i \tau / n) \) for \( n = 1, 2, 4, \frac{1}{2}, \frac{1}{4}, 8 \), respectively, and yielding the same j-function \( j(\tau) \), just like the three formulas for \( R(q) \). Thus finding 1) linear j-function \( n \)-tuplication formulas using the same continued fraction and 2) argument variants \( q \) using the same j-function are two sides of the same coin. It is not trivial to find \( n \) as the possible values of \( n \) are dependent on the modular group \( \Gamma(n) \). For the Rogers-Ramanujan continued fraction which is connected to \( \Gamma(5) \) we saw that it had \( q = \exp(2\pi i \tau / n) \) for \( n = 1, 5, 1/5 \).

So how do we find these variant formulas? One way can be seen in the fact that the variant formulas for \( R(q) \), namely \( s(\tau) \) and \( t(\tau) \), have identical coefficients though the latter involved a change of variable. The approach would be to have a basic or "template" formula, and using a modular relation between \( R(q) \) and \( R(q^n) \) for appropriate rational \( n \) generate a new one from the old. Ramanujan in his lost Notebook wrote five modular equations relating \( R(q) \) and \( R(-q) \), \( R(q^2) \), \( R(q^3) \), \( R(q^4) \), \( R(q^5) \). However, not all \( n \) will give a formula linear in \( j(\tau) \).

Another pair of formulas with identical coefficients, if one hasn’t noticed it already, are for \( j(z/2) \) and \( j(8z) \) using the second continued fraction. It seems nature really does not multiply entities unnecessarily as these coefficients appear yet again in the formula for the Ramanujan-Gordon-Gollnitz continued fraction, given by (in the form used by Selberg),

\[ v(\tau) = \frac{q^{1/2}}{1 + \frac{q + q^2}{1 + \frac{q^3}{1 + \frac{q^3 + q^6}{1 + \frac{q^8}{1 + \ldots}}}}} \]

then we have, (after a change of variable of \( v \to x \)),

\[ u(z) = \sqrt{-1 + \sqrt{2}} = \frac{\sqrt{2}e^{-2\pi/8}}{e^{-2\pi}} = 0.6435... \]
\[ j(\tau) = \frac{4(x^4 + 60x^3 + 134x^2 + 60x + 1)^3}{x(x + 1)^2(x - 1)^6} \]

where the variables \( v \) and \( x \) are related by,

\[ v^2 + v^{-2} = 4x + 2 \]

Compare to the relevant formulas of \( U(q) \). As eta quotients (see paper by Duke),

\[ v^2 + v^{-2} = \frac{\eta^4(\tau)\eta^2(4\tau)}{\eta^2(2\tau)\eta^4(8\tau)} + 6 \]

Example:

\[ v(\sqrt{-1}) = v = \frac{e^{-\pi}}{1 + \frac{e^{-2\pi} + e^{-4\pi}}{1 + \frac{e^{-8\pi}}{1 + \frac{e^{-12\pi}}{1 + \ldots}}}} = 0.04313... \]

where \( v \) is the appropriate root of,

\[ v^2 + v^{-2} = 4 \left( 3 + 2\sqrt{2} + 2\sqrt{4 + 3\sqrt{2}} \right)^2 + 2 \]

In summary, given a single \( j \)-function value, the continued fraction formulas have as one of their roots the evaluation of \( R(q) \) at three arguments \( q \), namely \( q = \exp(2\pi i \tau / n) \) for \( n = 1, 5, 1/5 \) and for \( U(q) \), for five arguments \( q = \exp(2\pi i \tau / n) \) for \( n = 1, 2, 4, \frac{1}{2}, \frac{1}{4} \). (If we limit ourselves to relations \( M \), then \( n = 8 \) shouldn’t count since it is already a quadratic.)

Conversely, given a single continued fraction value of \( R(q) \) or \( U(q) \), then one can find the \( j \)-function also at three and five arguments, respectively. We will see that our results extend to the next continued fraction as well.

II. Cubic Continued Fraction

For Ramanujan’s cubic continued fraction, \( C(q) \),

\[ C(q) = \frac{q^{1/3}}{1 + \frac{q + q^2}{1 + \frac{q^2 + q^4}{1 + \frac{q^3 + q^6}{1 + \ldots}}}} \]
which is connected to the modular group $\Gamma(6)$, we can find six forms of $C(q)$, call them $c_i(\tau)$, with $q = \exp(2\pi i \tau / n)$ for $n = 1, 2, 3, 6, 1/3, 1/6$ such that the modular relation with rational coefficients is only linear in $j(\tau)$. We will focus on the first four, as the last two can be derived from the first two.

**Case 1:**

For $n = 1$, let $c_1(\tau) = C(q)$ where $q = e^{2\pi i \tau} = \exp(2\pi i \tau)$. Then $c = c_1(\tau)$ is a root of the $j$-function formula

$$j(\tau) = \frac{(256c^{12} + 256c^9 + 960c^6 + 232c^3 + 1)^3}{c^3(c^3 + 1)^3(8c^3 - 1)^6} \quad (\text{eq. 1a})$$

In terms of the Dedekind eta function, it is given as the cubic modular relation,

$$\frac{\eta(\tau / 3)}{\eta(3\tau)}^3 + 3 = \frac{4c^3 + 1}{c} \quad (\text{eq. 1b})$$

Note that since this obviously has three roots, then one has to determine which of the roots $c$ actually corresponds to $c_1$, especially if all three roots are real. The rest of the cases also involve a cubic modular relation and a similar determination must be done for each case.

**Case 2:**

For $n = 2$, let $c_2(\tau) = C(q)$ where $q = e^{2\pi i \tau} = \exp(2\pi i \tau)$. Then $c_2(\tau)$ is a root of,

$$j(\tau) = -\frac{(16c^{12} - 464c^9 + 240c^6 - 8c^3 + 1)^3}{c^6(c^3 + 1)^6(8c^3 - 1)^3} \quad (\text{eq. 2a})$$

In terms of the eta function,

$$\frac{\eta(\tau / 3)}{\eta(3\tau)}^3 + 3 = \frac{1 - 2c^3}{c^2} \quad (\text{eq. 2b})$$

This is the form used by Bernd et al in their paper which we will discuss in the next section, though $c_2$ was expressed in terms of a certain function of Ramanujan’s.

**Case 3:**

For $n = 3$, let $c_3(\tau) = C(q)$ where $q = e^{2\pi i \tau / 3} = \exp(2\pi i \tau / 3)$. Then $c_3(\tau)$ is a root of,
\[
j(\tau) = \frac{(256c^{12} + 256c^9 - 8c^3 + 1)^3}{c^9(c^3 + 1)(8c^3 - 1)^2}
\]
(eq. 3a)

or,

\[
\left( \frac{\eta(\tau / 9)}{\eta(\tau)} \right)^3 + 3 = \frac{4c^3 + 1}{c}
\]
(eq. 3b)

**Case 4:**

For \( n = 6 \), let \( c_4(\tau) = C(q) \), where \( q = e^{\pi \tau / 3} = \exp(\pi i \tau / 3) \). Then \( c_4(\tau) \) is a root of,

\[
j(\tau) = -\frac{(16c^{12} + 16c^9 - 8c^3 + 1)^3}{c^{18}(c^3 + 1)^2(8c^3 - 1)}
\]
(eq. 4a)

or,

\[
\left( \frac{\eta(\tau / 9)}{\eta(\tau)} \right)^3 + 3 = \frac{1 - 2c^3}{c^2}
\]
(eq. 4b)

**Examples:**

For convenience, for our first four examples we will use a simple case of \( \tau \). Given \( d = 8 = 4^*2 \), a discriminant also with class number 1, so \( \tau = \sqrt{-2} \) and \( j(\sqrt{-2}) = 20^3 \). Substituting this value into (eq.1a), it factors into several polynomials. We then either numerically solve the cubic in \( c \) using (eq.1b) or numerically evaluate the continued fraction and we can determine the correct factor. Doing the same for the other three cases we find the following evaluations,

\[
c_1(\sqrt{-2}) = \frac{1 - \sqrt{6 - 3\sqrt{3}}}{2} = \frac{e^{-2\pi \sqrt{3}/3}}{e^{-2\pi \sqrt{3}/3} + e^{-4\pi \sqrt{3}}} = 0.0517...
\]

\[
c_2(\sqrt{-2}) = -2 + \sqrt{6} = \frac{e^{-\pi \sqrt{3}/3}}{e^{-\pi \sqrt{3}/3} + e^{-2\pi \sqrt{3}}} = 0.2247...
\]
\[
c_3(\sqrt{-2}) = \left(\frac{-1 - 2\sqrt{2} + \sqrt{3} + \sqrt{6}}{8}\right)^{1/3} = e^{-2\pi \sqrt{2}/9} + e^{-4\pi \sqrt{2}/9} = 0.3534...
\]

\[
c_4(\sqrt{-2}) = \left(\frac{-1 + \sqrt{3}/2}{2}\right)^{1/3} = e^{-\pi \sqrt{2}/9} + e^{-2\pi \sqrt{2}/9} = 0.4825...
\]

and so on for other values of the j-function. Thus, given just one value of the j-function, using the formulas we can find the exact evaluations of Ramanujan’s cubic continued fraction for at least four arguments \( q \).

For form \( \tau = (1 + \sqrt{-d})/2 \), let \( d = 3 \) also with class number 1, and we have \( \tau = (1 + \sqrt{-3})/2 \) and \( j(\tau) = 0 \). We then find that,

\[
c_1(\tau) = \frac{(-1 + 2^{1/3})(1 + \sqrt{-3})}{2(2^{2/3})} = e^{2\pi /3} = 0.0818... + 0.1418 I...
\]

where for clarity, \( k = \tau \sqrt{-1} \). Likewise, we have,

\[
c_2(\tau) = \frac{-2 + 4(2^{1/3}) - 2^{2/3} + (2 - 2^{2/3})\sqrt{-3}}{4} = 0.3630... + 0.1786 I...
\]

\[
c_3(\tau) = \frac{\left((-1 + 2^{1/3})(2 + 2^{1/3} + 2^{1/3} \sqrt{-3})\right)^{1/3}}{2} = 0.4935... + 0.0983 I...
\]

\[
c_4(\tau) = \left(\frac{-1 + 2^{1/3}}{2}\right)^{1/3} = 0.506528...
\]

While the first three cases evaluates to a complex value, surprisingly, the last one is real.

To summarize, we found four modular polynomials \( M_i \),
such that by substituting $c_1, c_2, c_3, c_4$ into their respective polynomials, we have four formulas for
the j-function at the same argument $\tau$. However, if we substitute any of the $c_i$ into any of the $M_i$,
what we get are values of the j-function at rational or integral multiples of $\tau$. We then can come
up with a sort of a multiplication table for the $c_i$ and the $M_i$ (though it’s more of an $n$-tuplication
table).

<table>
<thead>
<tr>
<th></th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>1</td>
<td>1/2</td>
<td>1/3</td>
<td>1/6</td>
</tr>
<tr>
<td>$M_2$</td>
<td>2</td>
<td>1</td>
<td>2/3</td>
<td>1/3</td>
</tr>
<tr>
<td>$M_3$</td>
<td>3</td>
<td>3/2</td>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>$M_4$</td>
<td>6</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

where the entries are the multiples of $\tau$. If the subscript of the $c_i$ and the $M_i$ match, then as
expected the entry is unity and we just have the $j(\tau)$ formulas given by eq.1a to 4a. It should be
explicitly pointed out that not just any of the three roots $c$ of the cubic equation involving eta
quotients will do. It has to be the appropriate one that corresponds to the continued fraction,
otherwise the table will not apply.

Example:

Given $\tau = \sqrt{-2}$ we know from our previous examples that $c_2(\tau) = (-2 + \sqrt{6}) / 2$.
Applying this same value to the four polynomials, it gives the j-function at the rational multiples
of $\tau$ indicated in the table, namely,

\[
\begin{align*}
  j(3\pi) &= 20^3 (5 + 2\sqrt{6})^2 (49 + 12\sqrt{6})^3, & j(\tau) &= 20^3 \\
  j(3\pi / 2) &= 20^3 (5 - 2\sqrt{6})^2 (49 - 12\sqrt{6})^3, & j(\tau / 2) &= 20^3 
\end{align*}
\]

III. Derivation

To derive our j-function formulas using the cubic continued fraction, we need a theorem
established by Berndt, Chan, and Zhang in their paper, “Ramanujan’s Class Invariants With
Applications To The Values Of q-Continued Fractions And Theta Functions". First we have to introduce certain functions.

Let the Ramanujan G-and g-functions, \( G_n \) and \( g_n \) be defined as,

\[
\prod_{k=1,3,5,\ldots}^{\infty} (1 + e^{-k\pi \sqrt{n}}) = 2^{1/4} e^{-\pi \sqrt{n}/2} G_n
\]

\[
\prod_{k=1,3,5,\ldots}^{\infty} (1 - e^{-k\pi \sqrt{n}}) = 2^{1/4} e^{-\pi \sqrt{n}/2} g_n
\]

They can also be expressed in terms of two of the three Weber functions \( f_i \),

\[
G_n = 2^{-1/4} f \sqrt{-n}
\]

\[
g_n = 2^{-1/4} f_1 \sqrt{-n}
\]

where the \( f_i \) in terms of eta quotients are,

\[
f(\tau) = \frac{e^{-\pi i /14}}{\eta(\tau + 1)} \frac{\eta(\tau + 1/2)}{\eta(\tau)}
\]

\[
f_1(\tau) = \frac{\eta(\tau / 2)}{\eta(\tau)}
\]

Note that Ramanujan limited the argument of his G-and g-functions to form \( \tau = \sqrt{-n} \) associated with even discriminants though the j-function formula we will derive from these functions can be extended to \( \tau = (1 + \sqrt{-d}) / 2 \) as well, which are associated with odd discriminants.

In Berndt’s et al paper, there were two relevant theorems that we can discuss, a noniplication formula giving \( G_n \) from \( G_6 \) (page 5, Theorem 2.1) and a formula for the cubic continued fraction \( c_2 \) (page 11, Theorem 3.4). For the former, a variable \( p \) was defined as,

\[
p = G_n^4 + G_{n^{-1}}
\]

which, as one can check, is essential to the first theorem. However, for the latter formula, I assume there was a typesetting slip and a second definition for \( p \) in terms of the g-function was more appropriate, given by,

\[
p = g_n^4 - g_{n^{-1}}
\]

One can see in the context of their examples that this may have been the intended definition. The example given was for \( n = 2 \), yielding \( g_n = 1 \), \( p = 0 \), and which will be true only for the latter definition for \( p \) and so we will use this.
With Ramanujan’s cubic continued fraction $C(q)$ as defined previously, and argument $q$ set as $q = e^{\pi \sqrt{-n}} = \exp(\pi i \sqrt{-n})$, then the formula was,

$$C(e^{-\pi \sqrt{n}}) = \sqrt{\frac{x - p}{y + p}} \left( \frac{p^2 + xy + 4}{2} - \sqrt{\frac{p^2 + xy + 2}{2}} \right)$$

where,

$$x = \sqrt{p^2 + 1}, \quad y = \sqrt{p^2 + 4}$$

and $p$ as the second definition. In hindsight, one can see that this is a particular case of our $c_2(\tau)$ for certain $\tau$.

Since we know that the Weber functions can be connected to the $j$-function, what we seek is to connect this formula to the $j$-function. First, let $c = C(e^{-\pi \sqrt{n}})$ and rationalizing the above formula, we find that,

$$p^2 = \frac{(2c^2 + 1)^2 (2c^2 + 4c - 1)^2}{4c(1 - 7c^3 - 8c^6)}$$

Second, we have the formula,

$$j(\tau) = \frac{(x - 16)^3}{x}$$

where the three roots $x_i$ are powers of the Weber functions,

$$x_i = f_1^{24}, -f_1^{24}, -f_2^{24}$$

then in terms of $f_1$, it is,

$$j(\tau) = \frac{(-f_1^{24} - 16)^3}{-f_1^{24}}$$

or since $g_n = 2^{-1/4} f_1$, then,

$$j(\tau) = \frac{(2^6 g_n^{24} + 16)^3}{2^6 g_n^{24}}$$

We now have three equations,

$$p = g_n^4 - g_n^{-4}$$
\[ p^2 = \frac{(2c^2 + 1)^2(2c^2 + 4c - 1)^2}{4c(1 - 7c^3 - 8c^6)} \]

\[ j(\tau) = \frac{(2^6 \cdot 8^{24} + 16)^3}{2^6 \cdot 8^{24}} \]

in the four unknowns \( p, g, j, c \). By eliminating \( p \) and \( g \) via resultants, we can have a single equation in the two unknowns \( j \) and \( c \), which is the connection we seek. Surprisingly, the final equation (disregarding multiplicity) is a quadratic in \( j \) though it factors with the roots given by,

\[ j(\tau) = \frac{(256c^{12} + 256c^9 + 960c^6 + 232c^3 + 1)^3}{c^3(c^3 + 1)^3(8c^3 - 1)^6} \]

\[ j(\tau) = -\frac{(16c^{12} - 464c^9 + 240c^6 - 8c^3 + 1)^3}{c^6(c^3 + 1)^6(8c^3 - 1)^3} \]

which corresponds to our formulas using \( c_1(\tau) \) and \( c_2(\tau) \). By eliminating the g-function, these are no longer limited to \( \tau = \sqrt{-n} \) but extend to generic \( \tau \). Looking at Table 1, the relationship between these two continued fractions is that they are either halving or duplication formulas (on each other) for the argument \( \tau \) of the j-function.

Given just one formula, we can derive the others if we have the modular relations between \( C(q) \) and \( C(q^3) \). Assuming \( c_1(\tau) \) with \( q = \exp(2\pi i \tau) \) as the “basic” formula, we can find \( c_2 \) using the relation between \( C(q) \) and \( C(q^3) \). To find \( c_3 \) and \( c_4 \), if one looks at Table 1 again, the entries for the \( c_1 \) and \( c_2 \) columns are triples of the entries for \( c_3 \) and \( c_4 \) respectively. So to derive the latter pair, we need the relation between \( C(q) \) and \( C(q^3) \).

The modular relations between \( C(q) \), \( C(-q) \), \( C(q^3) \), and \( C(q^5) \) have already been established by H.H. Chan. However, we may want a procedure that can be applied to any continued fraction. What we need then are the \( n \)-tuplication formulas for the \( j \)-function in terms of the Dedekind eta function. Such formulas are known for \( n \) where \( n-1 \) divides 24.

We can illustrate the procedure for the case \( n = 3 \). The triplication formula is given by,

\[ j(\tau) = \frac{(x + 3)^3(x + 27)}{x} \quad \text{(eq.5a)} \]

\[ j(3\tau) = k = \frac{(y + 3)^3(y + 27)}{y}, \text{ where } y = \frac{3^6}{x} \quad \text{(eq.5b)} \]

and \( x = \left( \frac{\eta(3\tau)\sqrt{3}}{\eta(\tau)} \right)^{12} \).

Since we also have the “template” formula using \( c_1 \), we now have three equations in the four variables \( x, k, j, c_1 \). By eliminating \( x \) between (eq.5a) and (eq.5b) we get a modular relation
between \( j = j(\tau) \) and \( k = j(3\tau) \). And by substituting the expression for \( j \) using \( c_1 \), we will have a final equation in \( k \) and \( c_1 \), and if it has a linear factor in \( k \), then it gives us a new formula. We do the same for \( c_2 \), and we get two j-function argument triplication formulas, one using \( c_1 \),

\[
j(3\tau) = k_1 = \frac{(256c_1^{12} + 256c_1^9 - 8c_1^3 + 1)^3}{c_1^9(c_1^3 + 1)(8c_1^3 - 1)^2}
\]

and another using \( c_2 \),

\[
j(3\tau) = k_2 = \frac{(16c_2^{12} + 16c_2^9 - 8c_2^3 + 1)^3}{c_2^{18}(c_2^3 + 1)^2(8c_2^3 - 1)}
\]

Of course, by letting \( c_1 = c_3 \) or \( c_4 \), then they also are the third and fourth formulas for \( j(\tau) \).

For \( n = 3 \), the final equation will have a linear factor for \( C(q) \) but not for \( R(q) \) nor \( U(q) \). One can see that the method can be applied to any continued fraction formula even without knowing the modular relations between \( (q) \) and \( (q^3) \) or in general, for any \( n \) such that \( n-1 \) divides 24.

Technically, the final equation had another factor linear in \( k \) for both cases, giving the fifth and sixth formulas, though they were rather complicated expressions. However, as was pointed out, given a “template” formula and a modular relation between \( (q) \) and \( (q^n) \), we can derive others. Thus, if we know this relation, we can express the two linear factors of the final equation in terms of this template. For the first case, we have,

\[
j(n\tau) = \frac{(256x^{12} + 256x^9 + 960x^6 + 232x^3 + 1)^3}{x^3(x^3 + 1)^3(8x^3 - 1)^6}
\]

with relations,

\[
c_1^3 = \frac{x(x^2 - x + 1)}{4x^2 + 2x + 1} \quad \text{(eq.6a)}
\]

\[
x^3 = \frac{c_1(c_1^2 - c_1 + 1)}{4c_1^2 + 2c_1 + 1} \quad \text{(eq.6b)}
\]

If the template has \( x = c_1 \), then \( n = 1 \). But if \( x \) is eliminated between it and (eq.6a), then we get \( n = 3 \), equivalent to the triplicating formula \( k_1 \) given earlier. If we use the other relation (eq.6b), then \( n = 1/3 \), and we have the second linear factor which is a trisecting formula. A similar argument applies for the second case. The second case in turn can be derived from the first using the relation between \( (q) \) and \( (q^2) \) since it is just \( n = 2 \), though deriving \( n = 1/2 \) results in a formula quadratic in \( j(\tau) \) and hence is not a relation \( M \) as we defined it.

One can also use a variation of the method described here to find modular relations between \( (q) \) and \( (q^n) \). Using the modular relation between \( j(\tau) \) and \( j(n\tau) \), we substitute for \( j(\tau) \) the formula for, say, \( c_1 \) using a variable \( x \), and for \( j(n\tau) \) the same formula but using a variable \( y \). The
single equation in \( x \) and \( y \) should factor and one factor will give the relation between \( (q) \) and \( (q^n) \). For example, for \( n = 4 \), the modular relation between \( x = C(q) \) and \( y = C(q^4) \) is given by,

\[
(x - y)^4 = (1 - 8x^4)(y + y^4)
\]

However, since one has to deal with polynomials of high degree, in practice it works best for small \( n \) and other methods should be used for larger \( n \).

**IV. Connection To Tetrahedron**

In [1] we gave various formulas to the \( j \)-function using Dedekind eta quotients. For order 9, we had,

**Form 1,**

\[
j(\tau) = \frac{y^3 (y^3 - 24)^3}{y^3 - 27}
\]

\[
y_1 = \left( \frac{\eta(\tau / 9)}{\eta(\tau)} \right)^3 + 3 \quad y_2 = \left( \frac{3\eta(9\tau)}{\eta(\tau)} \right)^3 + 3
\]

**Form 2,**

\[
j(\tau) = \frac{z^3 (z^3 + 216)^3}{(z^3 - 27)^3}
\]

\[
z = \left( \frac{\eta(\tau / 3)}{\eta(3\tau)} \right)^3 + 3
\]

The roots of form 1 belong to a family of functions which we called \( k \)-function and for this particular order, it is associated with the modular group \( \Gamma(9) \). For form 2 though, technically these are not covered by those functions.

We also discussed the three polyhedral groups and found that the icosahedral and the octahedral groups had a continued fraction associated with it. So perhaps the last group, the tetrahedral, also has one. Recall that to see these connections we have to use the polyhedron’s projective geometry. Given a tetrahedron with unit circumradius, if we project its vertices using a stereographic projection and expressing these vertex locations as roots of an equation with a variable \( z \), we get,

\[
P(z,1) = z^4 - 2\sqrt{2}z = 0
\]

If with unit inradius, the face center positions are,

\[
H(z,1) = 2\sqrt{2}z^3 + 1
\]
Given the j-function formula of order 9, form 2,

\[ j(\tau) = \frac{z^3(z^3 + 216)}{(z^3 - 27)^3} \]

if we transform that by letting \( z = -3\sqrt{2}x \), then,

\[ j(\tau) = -\frac{1728(x^4 - 2\sqrt{2}x)^3}{(2\sqrt{2}x^3 + 1)^3} \]

If the coefficients are familiar it is because they are identical to the coefficients of the equations for the tetrahedron. Note that the formula has exactly 12 roots and the tetrahedral group is of order 12. We observed similar relationships for the other two polyhedral groups.

Since we now have formulas for the j-function using \( C(q) \), what we do is simply **equate these to the j-function formulas using eta quotients of appropriate order**, see if it factors and if there is a linear relationship. For \( c_1 \), if we equate it to order 9, form 2,

\[ \frac{z^3(z^3 + 216)}{(z^3 - 27)^3} = \frac{(256c^{12} + 256c^9 + 960c^6 + 232c^3 + 1)^3}{c^3(c^3 + 1)^3(8c^3 - 1)^6} \]

we get two, though we choose the simpler one given by,

\[ 1 + 4c^3 - cz = 0 \]

Solving this cubic in \( c \), we have the pretty solution,

\[ c_1 = \frac{\left(-1 + \sqrt{1 - x^3}\right)^{1/3} + \left(-1 - \sqrt{1 - x^3}\right)^{1/3}}{2} \]

where,

\[ x = \frac{1}{3} \left( \frac{\eta(\tau/3)}{\eta(3\tau)} \right)^3 + 1 \]

though it must be pointed out again that if there are three real solutions, then the correct root \( c \) should be properly determined. The same form equated to \( c_2 \) also results in a linear relationship. However, for \( c_3 \) and \( c_4 \), we have to equate it the form 1 version.

Thus, **there is** a relationship between Ramanujan’s cubic continued fraction \( C(q) \) and the tetrahedron. A bit tenuous, perhaps, but still there.

**V. Conclusion**
In this paper, the primary objective was to discuss various j-function formulas using continued fractions, in particular $C(q)$, and to propose a connection between $C(q)$ and the tetrahedral group.

If indeed there is a connection, it is amazing that it was just one man, Ramanujan, who found the continued fractions associated with all three polyhedral groups, though it is granted that he probably did not know of this aspect of this work. How he came up with them is another matter entirely. While one can arbitrarily define a continued fraction obeying certain rules, I do not suppose that was how he found them, considering the particular ones we have discussed have connections to group theory and modular functions.

One thing can be mentioned though. The connection between $C(q)$ and the tetrahedron doesn’t seem to be as direct as those for the other relationships. Since the fourth continued fraction we discussed, the Ramanujan-Gordon-Gollnitz $V(q)$, is connected to $U(q)$ (there is in fact a quadratic modular relation between them) it is then also connected to the octahedron. Thus several kinds of continued fractions can be indirectly connected to a polyhedral group.

There were other $q$-continued fractions that Ramanujan investigated. And there are certainly more waiting to be discovered. It is possible there might be one that is more directly connected to the tetrahedron than $C(q)$ is. How to find it though is anyone’s guess.

--End--

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Titus Piezas III
May 21, 2005
tpiezasIII@uap.edu.ph ← (Remove “III” for email)
www.geocities.com/titus_piezas/ramanujan.html ← (Click here for an index)

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