“Euler’s Extended Conjecture and $a^k + b^k + c^k = d^k$ for $k > 4$”

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“We müssen wissen. Wir werden wissen.”
(“We must know. We shall know.”) – David Hilbert

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I. Introduction

We’ll start this paper on Euler’s Extended Conjecture (EEC) by giving our main result, that is, a four-parameter solution to the equation,

$$x_1^5 + x_2^5 + x_3^5 = x_4^5$$  \hspace{1cm} (eq.0)

which, if we disregard sign, is equivalent to the form $x_1^5 + x_2^5 = x_3^5 + x_4^5$. Let,

$$(\sqrt[p]{p} + \sqrt[q]{q})^5 + (\sqrt[p]{p} - \sqrt[q]{q})^5 = (\sqrt[r]{r} + \sqrt[s]{s})^5 + (\sqrt[r]{r} - \sqrt[s]{s})^5$$

then,

$$p = 5(bc-ad)(c^2+10cd+5d^2)^2$$
$$q = (a^2+10ab+5b^2)^3 - (ac+10bc+5bd)(c^2+10cd+5d^2)^2$$
$$r = 5(bc-ad)(a^2+10ab+5b^2)^2$$
$$s = -(c^2+10cd+5d^2)^3 + (ac+10ad+5bd)(a^2+10ab+5b^2)^2$$

for arbitrary $a,b,c,d$. As an example, let $a = 1$, $b = 9$, $c = 3$, $d = 3$ and we have,
\[-(2272\sqrt{22} + 288\sqrt{30})^5 + (2272\sqrt{22} + 288\sqrt{30})^5 = -(2528\sqrt{6} + 992\sqrt{30})^5 + (2528\sqrt{6} + 992\sqrt{30})^5\]

The author is aware of only one other formula for (eq.0), a two-parameter solution found by A. Desboves in 1880 and independently by N. Elkies in 1995 to be given later. It may in fact be just a special case of the new formula just given. The Desboves-Elkies solution is multigrade, valid for multiple exponents \( k = 1,2,5 \). The new formula, for \( a,b,c,d \) satisfying a certain condition, can also be true for \( k = 1,2,5 \). It may be the complete parametrization of these \textit{quintic quadruples} since it was derived using Euler’s method to find the complete one for cubic quadruples \( x_1^3 + x_2^3 = x_3^3 + x_4^3 \). If so, then to find a counterexample to EEC it suffices to find appropriate values such that the expressions \( p,q,r,s \) are all squares.

So what exactly is EEC? Well, our story starts a long time ago, in a land far away…

\section*{II. Historical Background}

In their article “\textit{Pythagoras’ Theorem In Babylonian Mathematics}”, J. O’Connor and E. Robertson mentions a Babylonian tablet from \textbf{c.1900 BC}, which has the following problem,

\begin{quote}
\textit{4 is the length and 5 the diagonal. What is the breadth?}
\textit{Its size is not known.}
\textit{4 times 4 is 16. 5 times 5 is 25.}
\textit{You take 16 from 25 and there remains 9.}
\textit{What times what shall I take in order to get 9?}
\textit{3 times 3 is 9. 3 is the breadth.”}
\end{quote}

We can infer that the Babylonians knew there were many more examples of two second powers equal to a second power and that \( 3^2 + 4^2 = 5^2 \) was not an isolated result. Another tablet, the famous Plimpton 322 also from the same time period and kept in Columbia University, contains various paired values in sexigesimal which can be seen as part of a Pythagorean triple. The largest pair, converted in decimal, is \( (18541, 12709) \) and a quick calculation shows that \( 18541^2 - 12709^2 = 13500^2 \). Quick for us using a calculator, but the size of this example shows the Babylonians must have known of a method to generate solutions to \( a^2 + b^2 = c^2 \) other than by randomly scribbling values on the sand.

(Incidentally, most of the modern results in equal sums of like powers were found by computer search. And interestingly enough, silicon chips in computers get their silicon mostly from, what else, sand. After all these years, from Archimedes reckoning on it to scientists using computers, mathematics still gets help from sand.)
Jumping a few millennia to the early 1630’s, a certain lawyer and “amateur” mathematician by the name of Pierre de Fermat (1601-1665) wrote on the margin of a copy of Diophantus’ *Arithmetica,*

> “…It is impossible for a cube to be the sum of two cubes, a fourth power to be the sum of two fourth powers, or in general for any number that is a power greater than the second to be the sum of two like powers. I have discovered a truly marvelous demonstration of this proposition that this margin is too narrow to contain.”” (Nagell, T., Introduction to Number Theory, p. 251-253)

Fermat’s Last Theorem, or FLT, in modern notation is that \(a^k + b^k = c^k\) has no non-trivial integral solution for \(k > 2\). This cryptic remark would tantalize generations of mathematicians and amateurs alike. (And in some quarters, still is.) If indeed he wrote this note when he first studied this book in the early 1630’s, then he would be around 30 years old at the time. By Sept 1636 he would mention the problem of two cubes whose sum is a cube in a letter to Sainte-Croix.

The next figure is the great Swiss mathematician Leonhard Euler (1707-1783). In one of his letters he wrote,

> “…It has seemed to many Geometers that this theorem [FLT] may be generalized. Just as there do not exist two cubes whose sum is a cube, it is certain that it is impossible to exhibit three biquadrates whose sum is a biquadrate, but that at least four biquadrates are needed if their sum is to be a biquadrate, although no one has been able up to the present to assign four such biquadrates. In the same manner it would seem to be impossible to exhibit four fifth powers whose sum is a fifth power, and similarly for higher powers.”” (Dickson, L., History of the Theory of Numbers, Vol 2, p. 648)

In a letter to Goldbach dated Aug 4, 1753, Euler claimed to have proven FLT for \(k = 3\). His conjecture above, now known as the *Euler’s Sum of Powers Conjecture* or simply ESC, that it would take at least \(k\) \(k\)th powers to sum to an \(k\)th power (other than the trivial identity \(x^k = x^k\)), must be after this letter. The first example of ESC for \(k > 3\) turned up only after more than 150 years, when R. Norrie finally found the four biquadrates in 1911, \[30^4 + 120^4 + 272^4 + 315^4 = 353^4\]

One reason why ESC is not so well-known is that the limelight was on FLT. Surprisingly, it also turned out to be false and is one instance that Euler was wrong. Lander, Parkin, and Selfridge in their seminal 1967 paper found that,

> “Since \(27^5 + 84^5 + 110^5 + 133^5 = 144^5\), then ESC is false.”

In their concluding remarks they also asked, given the diophantine equation \((k,m,n)\),

\[x_1^k + x_2^k + \ldots + x_m^k = y_1^k + y_2^k + \ldots + y_n^k\]
where \(x_i\) and \(y_i\) are assumed to be positive integers. Then,

[1] Is it true that \((k,m,n)\) is never solvable when \(m+n < k\)?

[2] For which \(k, m, n\) such that \(m+n = k\) is \((k,m,n)\) solvable?

[3] Is \((k,m,n)\) always solvable when \(m+n > k\)?

To quote further, “...The results presented in this paper tend to support an affirmative answer to [1]. Question [2] appears to be especially difficult. The only solvable cases with \(m+n = k\) known at present are \((4,2,2), (5,1,4),\) and \((6,3,3)\)” (Lander, L., Parkin, T., and Selfridge, J., “A Survey of Equal Sums of Like Powers”, Math. Of Computation, Vol 21, 1967.)

This author made minor changes in the numbering of questions but otherwise they are quoted verbatim. Also, from this point onwards we will adopt the standard notation in the literature of \((k,m,n)\) with \(k\) as the power but with the modification that \(m\) is the number of terms on one side, and \(n\) on the other. Thus, the label \((4,1,3)\) equally describes,

\[a^4 + b^4 + c^4 = d^4, \quad \text{or} \quad a^4 = b^4 + c^4 + d^4\]

since mathematically they are equivalent anyway.

Some comments: First, there is a related problem to the one discussed by Lander et al, namely the Prouhet-Tarry-Escott Problem (PTE). However, this seeks for a multigrade solution. Furthermore, it usually involves a so-called “balanced equation”, with an equal number of terms on both sides. We are more after a minimal number of terms for \((k,m,n)\) and solutions need not be multigrade. For more on PTE see Chen Shuwen’s excellent site hosted at http://euler.free.fr/eslp/eslp.htm.

Second, three more cases of \(k = m+n\) are now known, namely \((4,1,3), (5,2,3),\) and \((8,3,5)\). Third, the remark “...The results presented in this paper tend to support an affirmative answer to [1]” is very cautiously worded, but if we define a conjecture as “...a proposition which is consistent with known data, but has neither been verified nor shown to be false” then while it does not make a definite stand the authors couldn’t quite resist making a leading statement. (Compare to Euler’s confident “It is certain that it is impossible...” with only the data for \((2,1,2)\) and \((3,1,3)\) known to him. Then again, he was Euler.)

So with the known result that ESC was wrong plus question [1], Ekl in a follow-up “New Results In Equal Sums Of Like Powers” (1998), then defined Euler’s Extended Conjecture (EEC). Just like FLT, this can be simply stated:

“The diophantine equation \((k,m,n),\)

\[x_1^k + x_2^k + \ldots + x_m^k = y_1^k + y_2^k + \ldots + y_n^k\]

has no integral solution for \(k > m+n,\) other than the trivial case when all \(x_i = y_i.\)"
Since this is a very broad conjecture, extending infinitely in two directions, namely the number of terms and the exponent, it is convenient to label it as cases by the number of terms $m+n$. The first two cases $m+n = 1$ and $m+n = 2$ are trivial and it only the third that is the first non-trivial case, $m+n = 3$ or $EEC(k,1,2)$, and is the familiar claim that $a^k + b^k = c^k$ has no solutions for $k > 3$ which is true by FLT. Hence, FLT is just a special case of EEC, up to a point. A proof of the whole of EEC would automatically imply FLT other than $k = 3$, which can be proven separately anyway as Euler did. The fourth case is $m+n = 4$, with two versions, $EEC(k,1,3)$ and $EEC(k,2,2)$,

$$a^k + b^k + c^k = d^k$$

and,

$$a^k + b^k = c^k + d^k,$$

which asserts that these have no solutions for $k > 4$. One can see the difference with FLT is that the greater number of terms mean we can “move” them around in various versions and which spell complications for proving the general conjecture. The separation into $m$ and $n$ terms is obviously for the case of even exponents since, if we drop the condition that $x_i$ and $y_i$ are assumed to be positive integers (which, mathematically, is rather arbitrary), then for odd exponents the various versions are equivalent to the most symmetrical form,

$$x_1^k + x_2^k + \ldots + x_n^k = 0$$

and to prove the conjecture for a particular odd exponent it suffices to prove it for one version. For the equations $(k,2,2)$ and $(k,1,3)$ where $k = 4$, we have the first solutions,

$$59^4 + 158^4 = 133^4 + 134^4 \quad (\text{Euler, c.1750})$$

$$2,682,440^4 + 15,365,639^4 + 18,796,760^4 = 20,615,673^4 \quad (\text{Elkies, 1986})$$

where the latter is the second counter-example to ESC, but the first for $k = 4$. A smaller one was later found by J. Frye and more examples were subsequently found. However, whether these two have solutions for $k > 4$ is definitely an open question, as well as for the fifth $m+n = 5$ with versions $EEC(k,1,4)$ and $EEC(k,2,3)$, and so all for an infinite number of cases.

The general objective of this paper is to elaborate more on EEC. In particular, since the first two cases are trivial and the third reduces to the known result for FLT, then it will deal with the fourth, namely $m+n = 4$, the first unknown and only its first exponent $k = 5$. This has two versions but since we are dealing with an odd power and as was pointed out we won’t distinguish between positive and negative solutions, then we will consider our result as applicable for both. For convenience, we will use the balanced equation $(5,2,2)$. The next exponent and its two versions, as well as all subsequent exponents, will be left for others.

It took about 200 years from the time Fermat sent the letter discussing FLT for $k = 3$ to the complete solution of $k = 5$ by Germain, Legendre, and Dirichlet in 1825. (Fermat himself proved the case for $k = 4$.) The next few decades saw more exponents $k$ being proven. Likewise, it took about 200 years from the time Euler stated his original conjecture to the discovery of the counter-example $27^5 + 84^5 + 110^5 + 133^5 = 144^5$ in 1967. Let us hope that the next few decades will also spell progress for EEC.
III. Derivation of $a^k + b^k = c^k + d^k$, for $k = 3$ and $5$

Before we derive the formula given in the Introduction, the one found by Desboves and Elkies is, let,

$$f_1 = x^2 + (\sqrt{2})xy - y^2,$$
$$f_2 = ix^2 - (\sqrt{2})xy + iy^2, \quad f_3 = -x^2 + (\sqrt{2})xy + y^2, \quad f_4 = -ix^2 - (\sqrt{2})xy - iy^2,$$

Define $E_k = f_1^k + f_2^k + f_3^k + f_4^k$, then $E_k = 0 \quad$ for $k = 1, 2, 5$.

Since $i$ is the imaginary unit $\sqrt{-1}$, then we can’t use this to find a counter-example to EEC. To find another parametrization, we will first show how Euler found the complete one for cubic quadruples as the quintic version was found by analogy. His method in fact was related to his proof of FLT for $k = 3$ which used the properties of the algebraic form $p^2 + 3q^2$.

(We’ve come across this form before, as the related version $a^2 + ab + b^2$ was prominent in the previous paper “Ramanujan and The Quartic Equation $2^4 + 2^4 + 3^4 + 4^4 + 4^4 = 5^4$.”) The method is as follows, given the quadruple,

$${x_1}^3 + {x_2}^3 = {x_3}^3 + {x_4}^3$$

let $x_1 = p + q, \; x_2 = p - q, \; x_3 = r + s, \; x_4 = r - s$, then,

$$p(p^2 + 3q^2) = r(r^2 + 3s^2) \quad \text{(eq.1)}$$

Euler noted that the second factors of each side have a common divisor of like form, call it $u^2 + 3v^2$. The objective then is to find expressions $p, q, r, s$ such that each side factors. He then gave such expressions as if plucked from thin air, assuming the gentle reader could follow. The author knew that this step was crucial, since one would have to find analogous expressions for the quintic case. It turned out one simply had to set,

$$p^2 + 3q^2 = (a^2 + 3b^2)(u^2 + 3v^2) \quad \text{(eq.2)}$$

and factor this over $\sqrt{-3}$. We then get two equations,

$$(p + q\sqrt{-3}) = (a + b\sqrt{-3})(u + v\sqrt{-3}) \quad \text{and}, \quad (p - q\sqrt{-3}) = (a - b\sqrt{-3})(u - v\sqrt{-3})$$

and solving for the two unknowns $p, q$ should enable us to express them in terms of $a, b, u, v$, given by,

$$p = au-3bv, \quad q = bu+av$$

Similarly for $r^2 + 3s^2$, we set,

$$r^2 + 3s^2 = (c^2 + 3d^2)(u^2 + 3v^2) \quad \text{(eq.3)}$$
and factoring over $\sqrt{-3}$ and solving for $r,s$ from the two resulting equations we find,

$$r = cu-3dv, \quad s = du+cv$$

Substituting $p,r$ and eq.2 and eq.3 into eq.1, we get,

$$(au-3bv)(a^2 + 3b^2)(u^2 + 3v^2) = (cu-3dv)(c^2 + 3d^2)(u^2 + 3v^2)$$

with the common divisor $u^2 + 3v^2$, so this is just linear in $u,v$, giving,

$$u(a^3+3ab^2-c^3-3cd^2) = 3v(a^2b+3b^3-c^2d-3d^3)$$

So, $u = 3(a^2b+3b^3-c^2d-3d^3)$ and $v = a^3+3ab^2-c^3-3cd^2$ (to eliminate denominators). Substituting these two into the expressions for $p,q,r,s$, we get,

$$(p+q)^3 + (p-q)^3 = (r+s)^3 + (r-s)^3$$

where,

$$p = 3(bc-ad)(c^2+3d^2), \quad q = (a^2+3b^2)^2 - (ac+3bd)(c^2+3d^2)$$
$$r = 3(bc-ad)(a^2+3b^2), \quad s = -(c^2+3d^2)^2 + (ac+3bd)(a^2+3b^2)$$

and we have the complete parametrization of cubic quadruples with four free variables $a,b,c,d$! Most of this is discussed in Dickson, p. 552-554. However, J. Binet gave a version (p.555) of Euler’s formula with just two variables,

$$x_1^3 + x_2^3 = x_3^3 + x_4^3$$

where,

$$x_1 = 1-(a-3b)(a^2+3b^2), \quad x_2 = (a+3b)(a^2+3b^2)-1$$
$$x_3 = (a+3b)-(a^2+3b^2)^2, \quad x_4 = (a^2+3b^2)^2-(a-3b)$$

and stated there was no loss of generality, though the justification for this assertion was not explicitly given by Dickson.

We now go into quintic quadruples by just following Euler’s footsteps, but with a small difference. Given,

$$x_1^5 + x_2^5 = x_3^5 + x_4^5$$

let, $x_1 = \sqrt{p+q}$, $x_2 = \sqrt{p-q}$, $x_3 = \sqrt{r+s}$, $x_4 = \sqrt{r-s}$, then,
\[(\sqrt{p})(p^2+10pq+5q^2) = (\sqrt{r})(r^2+10rs+5s^2) \quad (\text{eq.4})\]

Like Euler, we consider the second factors as having a common divisor. We set,
\[p^2+10pq+5q^2 = (a^2+10ab+5b^2)(u^2+10uv+5v^2) \quad (\text{eq.5})\]
and factor this over \(\sqrt{5}\) to get two equations just like before. We then solve for \(p,q\) which are given by,
\[p = au-5bv, \quad q = bu+av+10bv\]

Similarly,
\[r^2+10rs+5s^2 = (c^2+10cd+5d^2)(u^2+10uv+5v^2) \quad (\text{eq.6})\]
so, \[r = cu-5dv, \quad s = du+cv+10dv\]

Substituting \(p,r\) and (eq.5) and (eq.6) into the square of (eq.4), we get,
\[(au-5bv)((a^2+10ab+5b^2)(u^2+10uv+5v^2))^2 = (cu-5dv)((c^2+10cd+5d^2)(u^2+10uv+5v^2))^2\]
and find (after simplification) \(u,v\) as,
\[u = 5(b(a^2+10ab+5b^2)^2 - d(c^2+10cd+5d^2)^2), \quad v = a(a^2+10ab+5b^2)^2 - c(c^2+10cd+5d^2)^2\]

Substituting these two into the expressions \(p,q,r,s\), we get,
\[(\sqrt{p} + \sqrt{q})^5 + (\sqrt{p} - \sqrt{q})^5 = (\sqrt{r} + \sqrt{s})^5 + (\sqrt{r} - \sqrt{s})^5\]
where,
\[p = 5(bc-ad)(c^2+10cd+5d^2)^2, \quad q = (a^2+10ab+5b^2)^3 - (ac+10bc+5bd)(c^2+10cd+5d^2)^2\]
\[r = 5(bc-ad)(a^2+10ab+5b^2)^2, \quad s = -(c^2+10cd+5d^2)^3 + (ac+10ad+5bd)(a^2+10ab+5b^2)^2\]
as given in the Introduction. Note that this is vaguely reminiscent of the formula for cubics. The big question is then: \textit{Is this the general parametrization for quintic quadruples?} After all, we essentially used the same steps. \textit{If} it is, and \textit{if} it can be proven that \(p,q,r,s\) can never be simultaneously squares (or if they can be but only for trivial \(x_i = y_i\)), then we can prove EEC(5,2,2)! A small step, but one has to start somewhere. Note that \(p,r\) can easily be made into squares by setting \(5(bc-ad) = v^2\) and solving for any of the free variables, say, \(d\). The problem is that by substituting this value for \(d\) into \(q,s\) we end up with two polynomials of degree 10 and 12 in four variables \((a,b,c,v^2)\) which simultaneously must be made into squares! Whether one can find such values is uncertain.
Earlier it was pointed out that just like the Desboves-Elkies formula, we can make our parametrization solve the multi-grade \( x_1^k + x_2^k = x_3^k + x_4^k \) for \( k = 1,2,5 \). To illustrate this, define,

\[
R_k = (\sqrt{p} + \sqrt{q})^k + (\sqrt{p} - \sqrt{q})^k + (-\sqrt{r} - \sqrt{s})^k + (-\sqrt{r} + \sqrt{s})^k
\]

then for \( k = 1,2,5 \),

\[
R_1 = 2(\sqrt{p} - \sqrt{r}), \quad R_2 = 2(p+q+r+s), \quad R_5 = 2(p^2+10pq+5q^2)\sqrt{p-2(r^2+10rs+5s^2)}\sqrt{r}
\]

Using the formulas established for \( p,q,r,s \), we find that,

\[
R_1 = 2(a^2+10ab+5b^2-c^2-10cd-5d^2)\sqrt{(bc-ad)},
\]

\[
R_2 = 2(a^2+10ab+5b^2-c^2-10cd-5d^2) P(z)
\]

where \( P(z) \) is a polynomial of degree 4. Obviously, \( R_5 = 0 \). Thus to set \( R_1 = R_2 = R_5 = 0 \), one simply has to find values \( a,b,c,d \) such that,

\[
a^2+10ab+5b^2 = c^2+10cd+5d^2
\]

which can easily be found.

Before we go to the next section, some interesting incidental facts about the parametrizations for cubic and quintic quadruples can be mentioned. **First**, \( \mathbb{Q}\sqrt{-3} \) is the smallest imaginary quadratic field with class number 1 while \( \mathbb{Q}\sqrt{5} \) is the smallest real quadratic field also with class number 1. **Second**, to recall Euler used the algebraic properties of the form \( F_1 = p^2 + 3q^2 \) to find the formula for \( k = 3 \). With a small linear substitution \( p = (-a+b)/2 \) and \( q = (a+b)/2 \), we get \( F_1 = a^2+ab+b^2 \). For \( k = 5 \), we used the form \( F_2 = p^2+10pq+5q^2 \). Letting \( p = (a-2b)/4 \) and \( q = (a+2b)/4 \), we get \( F_2 = a^2+ab-b^2 \) and which differs from the previous only by a sign, yet it makes all the difference as the discriminant \( d \) of the former is \( d = -3 \) while the latter is \( d = 5 \). **Third**, the algebraic form \( F_1 \), much discussed in a previous paper, also appears in a geometric context (perhaps not surprisingly) in the formula for the volume of the triangular and square pyramidal frustums. The general pyramidal frustrum is simply a pyramid with a base of \( n \) sides and the top chopped off. For \( n = 3, 4 \), the volume formulas are,

\[
V_3 = (a^2+ab+b^2)(h/12)\sqrt{3}
\]

\[
V_4 = (a^2+ab+b^2)(h/3)
\]

where \( a \) is the top side length, \( b \) is the base side length, and \( h \) is the height (with the assumption that top and base are regular \( n \)-gons). It should be interesting to know if the form \( F_2 \) appears somewhere as a formula, perhaps involving pentagons. **Fourth**, mirroring the coefficients of \( F_2 = p^2+10pq+5q^2 \), the equation,
\[ x^4 - 10x^2 + 5 = 0 \]

is connected to the tangent function, having the solutions \( \tan(\pi n/10) \) for \( n = 1, 2, 3, 4 \) given explicitly by,

\[
\begin{align*}
x_1 &= (1/5)\sqrt{(25-10\sqrt{5})}, & x_2 &= \sqrt{(5-2\sqrt{5})}, & x_3 &= (1/5)\sqrt{(25+10\sqrt{5})}, & x_4 &= \sqrt{(5+2\sqrt{5})}.
\end{align*}
\]

Finally, note that the Pythagorean formula is essentially the distance formula between two points \((x_1, y_1)\) and \((x_2, y_2)\) in the Euclidean plane. This can be generalized to \(n\)-space and for \( n = 3 \), the analogous one for two points \((x_2, y_2, z_2)\) and \((x_1, y_1, z_1)\) is,

\[
d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.
\]

so perhaps it may not be surprising that Euler’s parametrization of \( x_1^k + x_2^k + x_3^k = x_4^k \) for \( k = 3 \) depended on the algebraic properties of \( a^2 + ab + b^2 \) which was connected to the volume formula of certain 3-dimensional objects.

**IV. Numerical Results**

For this section, solutions for diophantine equations \((k,m,n)\) will be given. Since fortunately very extensive results are now known, we will restrict our list into three categories corresponding to the questions asked in Lander et al but with further restrictions, for: **Type I** \((k > m+n)\), **Type II** \((k = m+n)\), and **Type III** \((k < m+n)\).

**A. Type I \((k > m+n)\)**

(No examples known)

**B. Type II \((k = m+n)\)**

For the sake of conciseness, this will be limited only to first results and \( k \) up to 10.

\[(3,1,2): \ (No \ solutions \ by \ Fermat’s \ Last \ Theorem)\]

\[(4,1,3): \ 2,682,440^4 + 15,365,639^4 + 18,796,760^4 = 20,615,673^4 \ (Elkies, \ 1986)\]

\[(4,2,2): \ 59^4 + 158^4 = 133^4 + 134^4 \ (Euler, \ c.1750)\]

\[(5,1,4): \ 27^5 + 84^5 + 110^5 + 133^5 = 144^5 \ (Lander, \ Parkin, \ 1967)\]

\[(5,2,3): \ 5027^5 + 6237^5 + 14068^5 = 220^5 + 14132^5 \ (Scher, \ Seidl, \ 1996)\]

\[(6,1,5): \]

\[(6,2,4): \]

\[(6,3,3): \ 3^6 + 19^6 + 22^6 = 10^6 + 15^6 + 23^6 \ (Rao, \ 1934)\]

\[(7,1,6): \]

\[(7,2,5): \]

\[(7,3,4): \]
As one can see, much work still needs to be done even for this partial list though at least we now know more than the previous generations.

C. Type III (k<m+n)

As this has the most extensive results, to make this list manageable it will be limited to the minimum possible difference, just a step above k=m+n, namely, k+1=m+n. These then would be the most difficult to find of this type but there are still many results known. To keep numbers down, we can just focus on k up to 11 and on two sub-types: the Eulerian (k,1,k) and the balanced equation (2n-1,n,n) for odd powers. (For even powers, (2n,n,n) is Type II.)

Eulerian solutions (k,1,k):

(2,1,2): \(3^2 + 4^2 = 5^2\)
(3,1,3): \(3^3 + 4^3 + 5^3 = 6^3\)
(4,1,4): \(30^4 + 120^4 + 272^4 + 315^4 = 353^4\) (Norrie, 1911)
(5,1,5): \(7^5 + 43^5 + 57^5 + 80^5 + 100^5 = 107^5\) (Sastry, 1934)
(6,1,6):
(7,1,7): \(127^7 + 258^7 + 266^7 + 413^7 + 430^7 + 439^7 + 525^7 = 568^7\) (Dodrill, 1999)
(8,1,8): \(90^8 + 223^8 + 478^8 + 524^8 + 748^8 + 1088^8 + 1190^8 + 1324^8 = 1409^8\) (Chase, 2000)
(9,1,9):
(10,1,10):
(11,1,11):

The case (6,1,6) is a bit odd. For k kth powers equal to a kth power, call this sum \(z^k\), one might assume that perhaps the larger the k, the larger would be z. However, Lander et al actually searched for this \((k = 6)\) below a certain bound and failed to find a solution. But this...
bound was z = 38300, much larger than Chase’s z = 1409 for k = 8! Either there was something wrong with the congruences they used (unlikely) or a programming or computer error (who knows) or it is really the case that if (6,1,6) does have a solution, then it will have z > 38300. Meyrignac’s Eulernet at http://euler.free.fr prominently has this equation as most wanted on its main page and they still have no solution (and the project’s been running since 1999) so they may have pushed this bound even higher.

**Balanced equations (2n-1,n,n):**

(3,2,2):  $1^3 + 12^3 = 9^3 + 10^3$ (Frenicle de Bessy, 1657; Ramanujan, 1919)
(5,3,3):  $24^5 + 28^5 + 67^5 = 3^5 + 54^5 + 62^5$ (Moessner, 1939)
(7,4,4):  $10^7 + 14^7 + 123^7 + 149^7 = 15^7 + 90^7 + 129^7 + 146^7$ (Ekl, 1996)
(9,5,5):  $26^9 + 30^9 + 91^9 + 101^9 + 192^9 = 12^9 + 17^9 + 116^9 + 175^9 + 180^9$ (Ekl, 1998)
(11,6,6):  

These type of equations are also interesting since for the equation pairs (3,2,2), (4,2,2) and (5,3,3), (6,3,3) parametric solutions are known. The question obviously is: does this generalize, or do (2n-1,n,n) and (2n,n,n) always have parametric solutions? The cases (7,4,4) and (9,5,5) were found by numerical search and the former in fact already has many examples, some of which found by A. Choudhry (2000) are even multigrade for k = 1,3,7. Whether these would turn out to be members of a parametric family is unknown.

In summary, there are 39 diophantine equations (k,m,n) in this admittedly arbitrary list. Out of the 39, less than half are known to have solutions. Some will undoubtedly fall in the next few years with the advent of faster computers, the Internet, and more people working on the field (most likely would be (8,4,4) and (10,5,5)). Moore’s Law in particular, which states that computing power doubles roughly every two years, makes the feasibility of a numerical search coupled with congruence restraints very reasonable for some (k,m,n). The author will maintain this list and update it as necessary. For more details, see the separate article “Timeline of Euler’s Extended Conjecture (EEC)”.

**V. Conclusion**

Almost 4000 years have passed since the ancient Babylonians knew that two second powers could sum to a second power and the time S. Chase found by computer the first example of eight eighth powers whose sum is an eight power. We now have many examples of k kth powers equal to a kth power, equations of such a high degree perhaps inconceivable to the Babylonians and definitely out of reach even if they could conceive of the possibility. While now we routinely discuss equations of any degree k, the solution of Eulerian (k,1,k) for higher powers (as “high” as k=6) still eludes us, and for anyone interested in diophantine equations this not knowing is as irritating as a pebble in the shoe. *We must know.*³
However, we also now know that three fourth powers can sum to a fourth power, as well as that the sum of four fifth powers can be a fifth power. But similar equations for the sixth? Or seventh, and so on?

Euler’s extended conjecture (EEC) generalizes these questions and asserts that \( m \) \( k \)th powers equal to \( n \) \( k \)th powers have no non-trivial solution for \( k > m+n \) and it is hoped this paper gave a clear introduction to the topic. After all, it was its objective, as well as a small contribution to the fourth case \( m+n = 4 \) for \( k = 5 \) by giving a new parametrization. Some questions, basically versions of the ones asked in Lander et al, are worth pointing out again, namely:

[1] Is \( k = m+n \) always solvable?
[2] Is \( (k,1,k) \) always solvable?
[3] Are \( (2n-1,n,n) \) and \( (2n,n,n) \) always parametrically solvable?

as well as a new one focused on this paper,

[4] Is the new solution for \( (5,2,2) \) the complete parametrization?

If indeed EEC is true, then why? Why can’t, for example, be three 12\textsuperscript{th} powers equal to three 12\textsuperscript{th} powers, all distinct? There must be a reason forbidding such solutions. For comparison, Fermat’s investigations on generalizing \( a^k + b^k = c^k \) for \( k > 2 \) eventually lead, via the Taniyama-Shimura conjecture, to an interesting connection between modular forms and elliptic curves. So where will EEC lead us? Who knows, but it may also be to previously unknown and unexpected connections between different and seemingly unrelated mathematical objects.

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Footnotes:

1. Fermat is sometimes considered as an amateur mathematician. However, the word “amateur” is derived from the Latin “\textit{amare}”, which means “to love”, hence words like amorous, amicable, amigo, etc. The mathematics amateur then, etymologically speaking, is a lover of mathematics in a similar sense that a philosopher is a lover of wisdom (the Greek “\textit{philein}” also means “to love”). Thus, professional mathematicians who love the field can truly be called, paradoxically, as _amateurs_.

2. Hardy almost left a message as tantalizing as Fermat’s. This was about the famous Riemann Hypothesis. To quote de Sautoy, “…\textit{On a rough sea crossing fearing for his life, he sent a joke telegram saying that he had found a wonderful proof of the hypothesis. The ship, however, did not sink.}” Since Hardy did prove there were an infinite number of zeros on the critical line, imagine the “Did he really?” questions if the ship did sink. (See de Sautoy’s article link below.)
3. For Hilbert, what he really wanted to know was the proof of the Riemann hypothesis. He allegedly said that if he could wake from a thousand year sleep, his first question would be to ask whether anyone had proved it.

--End--

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